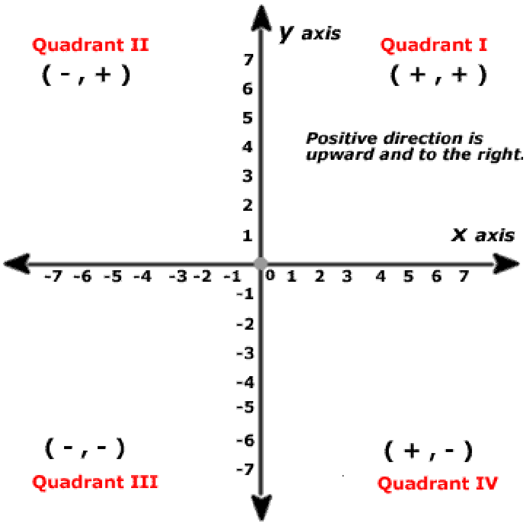
FUNCTIONS

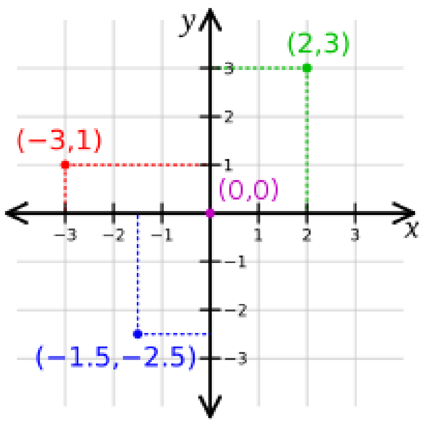
Coordinate system (Cartesian plane)

The French philosopher, mathematician and scientist René Descartes (17^{th} century) was one of the most important and influential thinkers in history. Descartes created the coordinate system we are going to study, which is also called the **Cartesian plane**.

The coordinate system is formed by two lines: A horizontal line that is called an **x-axis** and a vertical line that is called a **y-axis**. Both lines meet at a point called an **origin**. The distances from the origin towards either the right or upwards are positive, and the distances from the origin towards either the left or downwards are negatives.
The two axes divide the plane into four regions or **quadrants**.



The location of any point on the coordinate plane is denoted by using a pair of coordinates (x, y), where X is the horizontal distance from the origin and Y is the vertical distance from the origin.



Each coordinate of a point P is obtained by drawing a line through p perpendicular to the associated axis, finding the point q where that line meets the axis, and interpreting q as a number of that number line.

Illustration of a Cartesian coordinate plane. Four points are marked and labelled with their coordinates: (2, 3), (−3, 1), (−1.5, −2.5), and the origin (0, 0).

Exercises

1. Name the point that has the coordinates.

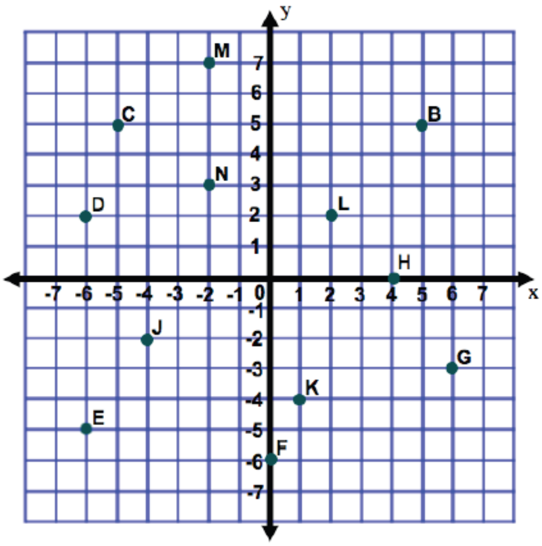
   a. (2, 2)　　b(-6, 2)
   c.(1,-4)　　d.(0,-6)
   e (-4,-2)

2. Write the coordinates of each point.

   a. B　　b. G　　c. E
   d. N　　e. H

3. In what quadrant is each point located?

   a. C　　b. J　　c. L
   d. M　　e. K



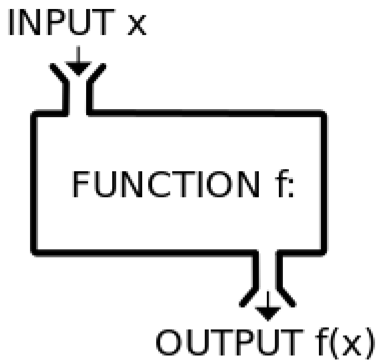
The concept of function

A **function** is a relation between two variables or magnitudes (x and y) where the value of the magnitude **y** depends on the value of the magnitude x.

X is called the **independent** variable.

Y is called the **dependent** variable.

Also, each value of x (or input) is related to exactly **one** output. This output is also denoted **f(x),** and is known as the **image** of x.



Example: Suppose a notebook costs €1.50 and we want to write the relationship of dependence between the number of notebooks we buy and the total cost that we pay for:

x = number of notebook we buy

y = f(x) = total cost we pay for

A function can be described in some different ways:

a) Using a formula: **f(x) = 1.5x**

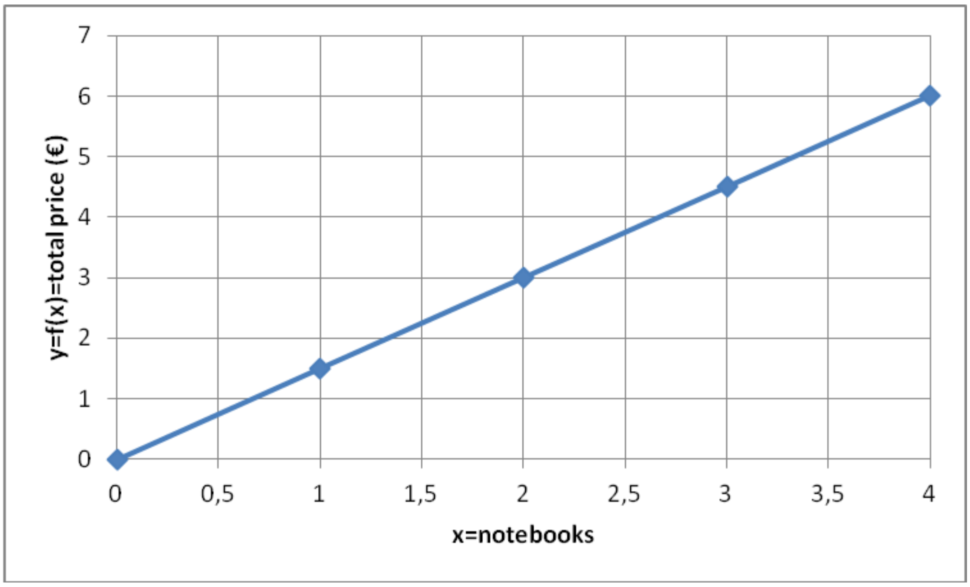
b) Using a table:

x	1	2	3	4
y	1.5	3	4.5	6

It means that when x=2, then y=3 (If you buy 2 notebooks, you pay €3.00), and so on.

c) Using a graph: It means drawing the function by plotting the pair of coordinates we obtained making the table:

The points obtained are: (1, 1.5), (2, 3), (3, 4.5), (4, 6)

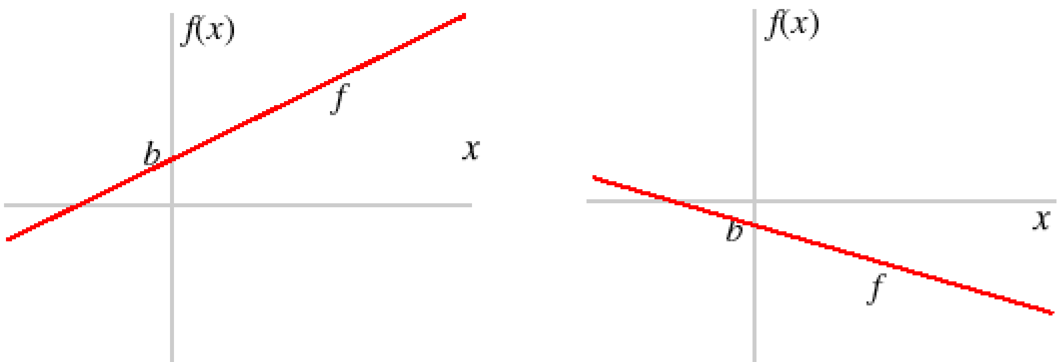


By studying the graph you see that the more notebooks you buy, the more money you pay; and you are able to see the values of y for each given value of x.

Linear functions

A linear function of a single variable has the form $\boldsymbol{f(x) = ax + b}$.

The graph for this function is a straight line with **slope** a; its value when $x = 0$ is b. Two examples are shown in the following figure.



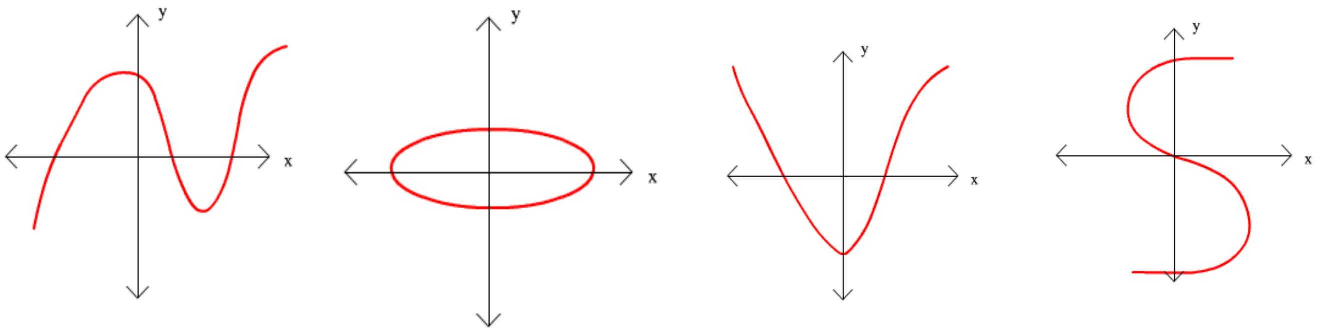
$f(x) = ax + b$ with $a > 0$ and $b > 0$ $\quad$ $f(x) = ax + b$ with $a < 0$ and $b < 0$

The function studied before is an example of a linear function, where the slope is 1.5 and b = 0.

Exercises

1. Indicate whether the relationship described is a function.
    a. Every person is assigned to his or her biological mother.
    b. Every mother is assigned to her children.
    c. Every word is assigned that is generated using a Google search.
    d. Every Hotmail account user name is assigned to the corresponding password.

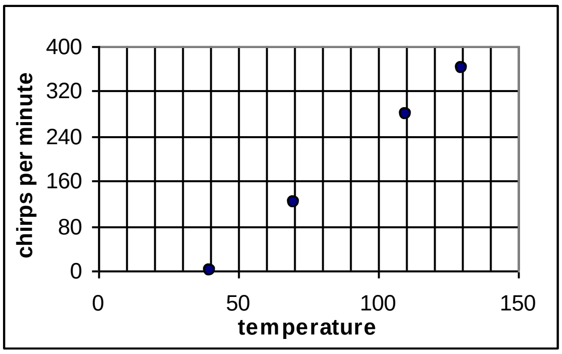
2. Determine whether the graph is a function and explain your reasoning.



3. When taking a taxi, we have to pay a fixed fee of €2.50 and €0.80 per kilometer.

    a. State the formula that represents the relationship between the number of kilometers and the final cost.
    b. If we go over 5 kilometers, how does it cost?
    c. Plot the graph.

4. The rate at which crickets chirp can be expressed as a function of temperature.



a)Find the equation for the number of chirps per minute as a function of temperature.
b)What is the average number of chirps per minute when the temperature is between 40 and 100 degrees?

Domain and range of a function

Definition of the Domain of a Function

The **domain** of a function **f(x)** is the set of all real numbers that variable **x** can take such that the value of the function is real.

Definition of the Range of a Function

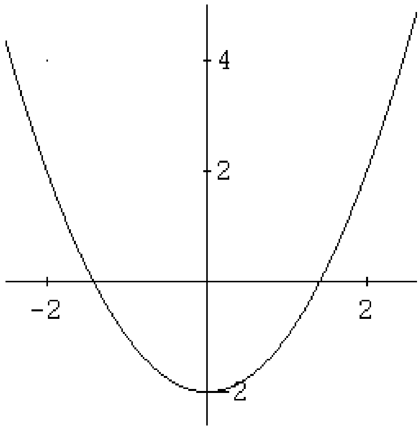
The **range** of f(x) is the set of all values that the function takes when x takes values in the domain.

Examples:

Example 0: Find the domain and the range of function f defined by:

$$f(x) = x^2 - 2$$

- Having a graph, you can find out the domain watching the values x takes on the x-axis. For discover the range, you have to focus on y-axis. Note the lowest point in the graph is y =2.
- So that: D = R and R = [-2 , +∞)



Finding out domain and range without having the graph.

- The domain of this function, as well as of any polynomial function, is the set of all real numbers, R. The range is the set of values that f(x) takes as x varies. If x is a real number, x^2 is either positive or zero. Hence we can write the following:
  $x^2 \geq 0$

- Subtract -2 to both sides to obtain
  $x^2 - 2 \geq -2$

- The last inequality indicates that $x^2 - 2$ takes all values greater that or equal to -2. The range of f is given by
  R = [-2 , +∞)

Example 1: Find the domain of function **f** defined by

$$f(x) = \frac{1}{x-1}$$

Solution to Example 1

- x can take any real number **except 1** since x = 1 would make the denominator equal to zero and the division by zero is not allowed in mathematics. Hence the domain in interval notation is given by

  D = (-∝, 1) U (1 , +∝) or by D = R - {1}

Exercise 1: Find the domain of function f defined by

f (x) = -1 / (x + 3)

Example 2: Find the domain of function f defined by

$$f(x) = \sqrt{2x - 8}$$

Solution to Example 2

- The expression defining function f contains a square root. The expression under the radical has to satisfy the condition
  2x - 8 ≥ 0 for the function to take **real** values.

- Solve the above linear inequality
  x ≥4

- The domain, in interval notation, is given by
  D = [4 , +∞)

Exercise 2: Find the domain of function f defined by:

$$f(x) = \sqrt{-x + 9}$$

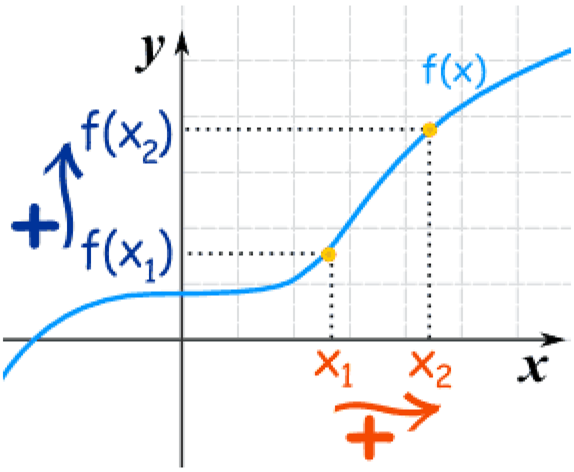
Continuity of a function

A real function *f* (*x*) is said to be *continuous* at a given interval if the function takes a real value for every value included in the interval. It means that the function is neither interrupted nor stopped over the interval.

Increasing and decreasing

a) Increasing function:

A function is "increasing" if the **y-value** increases as the **x-value** increases, like this:



It is easy to see that **y=f(x)** tends to go **up** as it goes **along**.

Using Algebra

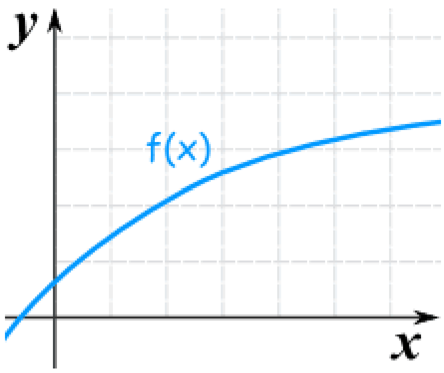
What if you can't plot the graph to see if it is increasing? In that case is is good to have a definition using algebra.

For a function **y=f(x)**:

when $x_1 < x_2$ then $f(x_1) \le f(x_2)$	**Increasing**
when $x_1 < x_2$ then $f(x_1) < f(x_2)$	**Strictly Increasing**

That has to be true for **any** x_1, x_2, not just some nice ones you choose.

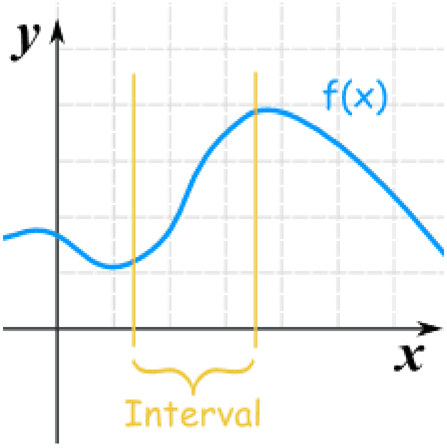
An Example:



This is also an increasing function
even though the rate of increase reduces

For An Interval

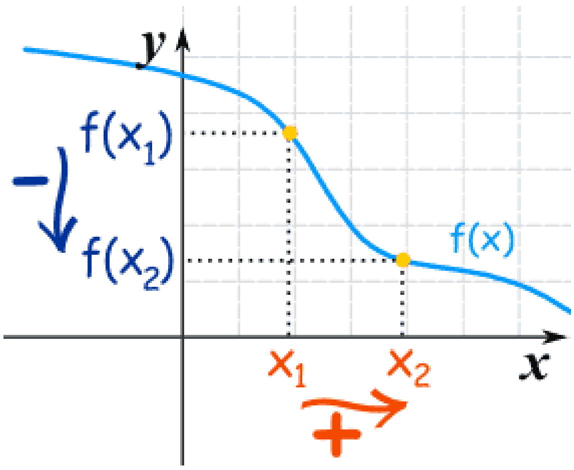
Usually you will only be interested in **some interval**, like this one:



This function is **increasing** for the interval shown
(it may be increasing or decreasing elsewhere)

b) Decreasing functions:

The **y-value** decreases as the **x-value** increases:



For a function **y=f(x)**:

when $x_1 < x_2$ then $f(x_1) \geq f(x_2)$	**Decreasing**
when $x_1 < x_2$ then $f(x_1) > f(x_2)$	**Strictly Decreasing**

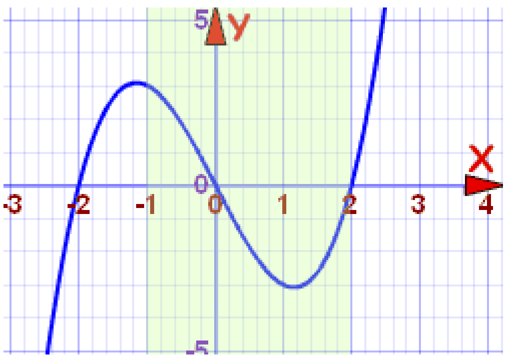
Notice that $f(x_1)$ is now larger than (or equal to) $f(x_2)$.

An Example

Let us try to find where a function is increasing or decreasing

Example: $f(x) = x^3-4x$, for x in the interval [-1,2]

Let us plot it, including the interval [-1,2]:



Starting from -1 (the beginning of the interval **[-1,2]**):

- at x = **-1** the function is decreasing,
- it continues to decrease until **about 1.2**
- it then increases from there, past x = **2**

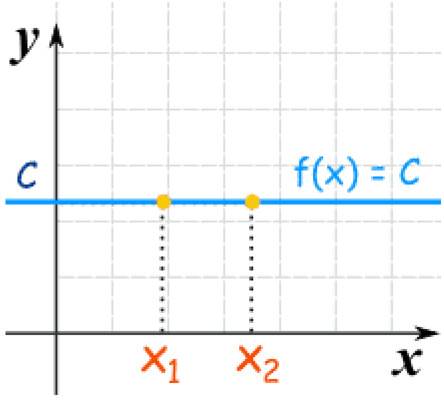
Without exact analysis we cannot pinpoint where the curve turns from decreasing to increasing, so let us just say:

Within the interval **[-1,2]**:

- the curve decreases in the interval **[-1, approx 1.2]**
- the curve increases in the interval **[approx 1.2, 2]**

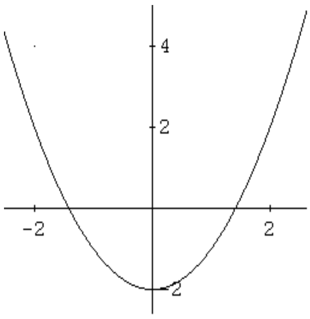
c) Constant Functions

A Constant Function is a horizontal line:



Maxima and minima

The **maximum** and **minimum** (plural: maxima and minima) of a function, are the largest and smallest value that the function takes at a point either within a given neighborhood (*local* or *relative* maximum or minimun) or on the function domain in its entirety (*global* or *absolute* maximum or minimun)

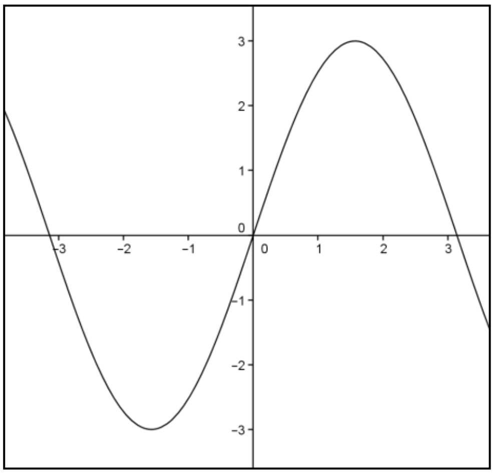


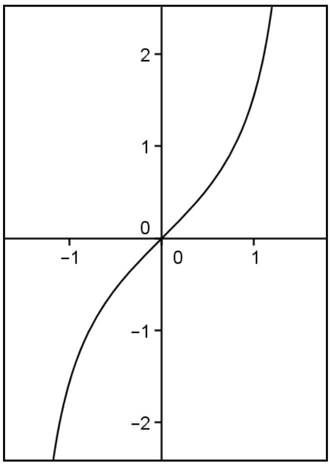
f(x) has a minimum at P(0, -2) and it has not maxima.

Symmetrical functions

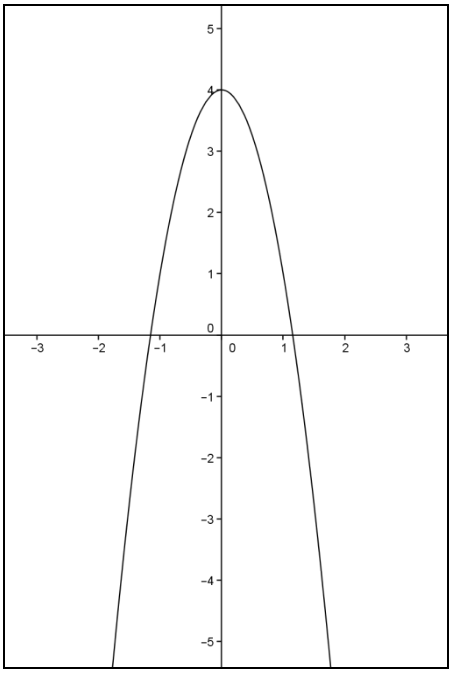
A function **f** is symmetric about y-axis or it is **even** when f(x) = f(-x) for every x belonging to the domain o of the function.

A function **f** is symmetric about origin or odd when f(-x) = - f(x) for every x belonging to the domain.

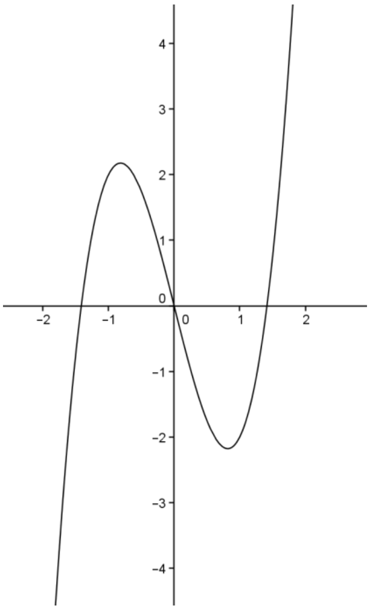




Determine algebraically whether $f(x) = -3x^2 + 4$ is even, odd, or neither.

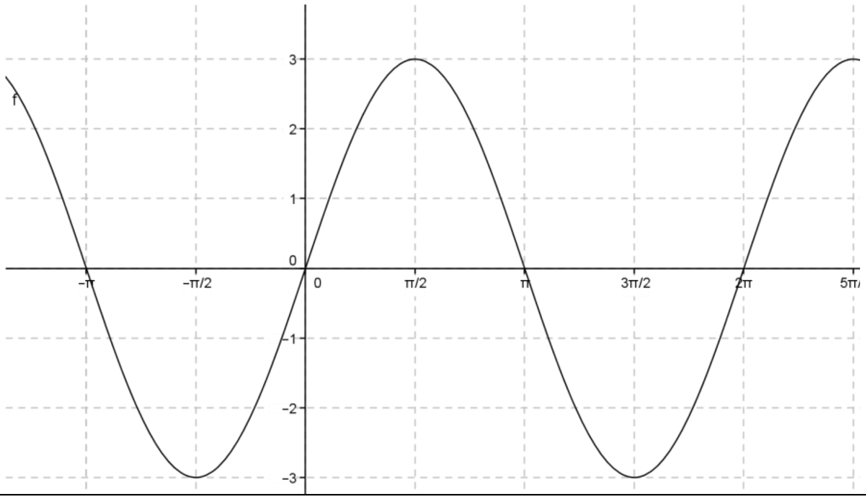


Determine algebraically whether *f*(*x*) = $2x^3 - 4x$ is even, odd, or neither.



Periodic functions

A **periodic function** is a function that repeats its values in regular intervals or periods. Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity.



y = 3sen (x)

T = 2π